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On the Fundamental Formulations of Electrodynamics

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VII. *On the Fundamental Formulations of Electrodynamics.*

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1. Modern electrical theory based on Maxwell's concept of an æthereal displacement current, is generally regarded as being sufficiently complete in itself to cover all actions so far revealed to us, if we exclude those intra-atomic phenomena which probably involve some additional but not necessarily inconsistent action in their working. There, still, however exists a good deal of uncertainty as to the actual results of the development of this theory in certain directions, and no account has yet been taken of the great degree of latitude allowed by it in its simplest and most general form. For example, in most presentations of the theory of energy streaming in the electromagnetic field the discussion is given in a way which might lead one to believe that Poynting's form* of the theory is the only one conceivable. A single alternative has on one occasion† been suggested, but rather as an improvement on Poynting's form than as an indication of its uncertainty. Whilst it cannot be denied that Poynting's theory is probably the most appropriate one yet formulated, yet it must be recognised that there are an infinite number of fundamentally different forms each of which is itself perfectly consistent with Maxwell's theory as expressed in his differential equations of electromagnetic interaction.

Again, but now we are on a different plane, it has usually been stated that Maxwell's theory is not of sufficient generality to cover the cases where there exists the complication of non-linear induction in ferromagnetic media.‡ This view appears to have originated with the idea that the *magnetic force* is the fundamental æthereal vector of the magnetic field, whereas, as a matter of fact, the only consistent view§ of the energy relations of such a field leads to the conclusion that the *magnetic induction* is the true æthereal vector, the magnetic force being an auxiliary vector

* 'Phil. Trans.,' A, vol. 175 (1884).
† Macdonald, 'Electric Waves,' Chs. IV., V., VIII.
‡ This is the view of practically all Continental writers on this subject.
§ *Cf.* 'Roy. Soc. Proc.,' A, vol. 93, p. 20 (1916).

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derived in the process of averaging the minute current whirls into their effective representation as a distribution of magnetic polarity.*

Further, the expression for the forcive of electromagnetic origin acting on the elements of a polarised medium still seems to be the subject of some doubt. MAXWELL† derives an expression in the magnetic case by statical considerations based on the method of energy, and then seems to regard it as generally valid under all circumstances. Objection has however been taken to MAXWELL'S expression by certain writers who, basing themselves on the presumed analogy between the dielectric and magnetic cases, prefer a form of expression differing from MAXWELL'S by a quantity which vanishes in the statical case considered but which is of fundamental importance in the derived problem of reducing the general forcive of electrodynamic origin to a representation by means of an imposed stress system. It appears in fact that the presence of this extra part in MAXWELL'S expression is effective in securing the ordinary expression for the subsidiary term arising in the induction, which has given rise to the conception of electromagnetic momentum, on account of its being a perfect time differential. In the alternative form of the theory the perfect time differential is not secured so that the idea of electromagnetic momentum is lost.‡ In his edition of MAXWELL'S treatise, J. J. THOMSON adds a note attempting to justify MAXWELL'S form of the expression, but his discussion can easily be shown to be erroneous, for he fails to distinguish between the true and complete currents of the theory, the latter containing a constituent, viz., the rate of change of æthereal displacement, which is not affected by the magnetic part of the complete electromagnetic forcive; nevertheless, the later discussions of the question from the point of view of the theory of electrons have confirmed MAXWELL'S original expression for the magnetic forcive, but they apparently still give the alternative expression for the dielectric case.

It was with the view to clearing up these and certain other difficulties that the present discussion was undertaken, the object aimed at being the formulation of a complete and precise statement of the theory in the only form in which it is logically consistent, then to compare this form with current statements of the theory,§ and finally to exhibit in their true aspects the various derived theories which are included in the general scheme. The original differential theory will be linked up with the subsequent dynamical theories by a discussion in its most general form of the derivation

* *Cf.* LARMOR, 'Roy. Soc. Proc.,' vol. 71 (1903), "On the Mechanical and Thermal Relations of the Energy of Magnetisation."

† 'Treatise,' II., Ch. II.

‡ *Cf.* LEATHEM, 'Roy. Soc. Proc.,' A, vol. 89 (1913), p. 34. In this note Mr. LEATHEM attempts to avoid the discrepancy by adding a new term to the force in the elementary polar theory. His only argument in favour of this force is that it overcomes the difficulty, so that it is not very convincing.

§ A complete statement of the fundamental results of the theory so far as they existed up till 1916 is given in my treatise, 'The Theory of Electricity' (Cambridge, 1918). The present paper may be regarded in some measure as a correction and generalisation of the statement there given.

of the fundamental equations based on the principle of Least Action, in the course of which certain inconclusive aspects of this derivation present themselves for consideration.

2. A complete statement of MAXWELL's theory as originally given and in the form which will include most of the recent extensions depends on certain field vectors which first require consistent and independent definition. These vectors :—

(*a*) E, the *electric force*, defined at any point of the field as the vectorial ratio to a small electric charge of the force acting on it when placed at rest in that position. When the point under consideration is inside the matter in the field there exists the possibility of an additional contribution to this force due to local conditions of the matter in the neighbourhood of the point, but for the present we shall disregard any complication of this kind.

(*b*) C, the complete electric current; in the most general case this consists of several distinct parts. Firstly, there is the differential drift of free ions constituting the true conduction current and the material dielectric displacement current; then there is a part due to the convection of charged and electrically polarised media, and finally the æthereal constituent essential and peculiar to MAXWELL's theory.

It has been definitely established that all but the last constituent of the current are in themselves true movements of electricity, or at least effectively equivalent to such, so that so long as we retain the definite concept of an electrical entity the origin of these different constituents merely remains a matter of kinematics, and they have, in fact, been fully dealt with on this basis.*

(*c*) H, the magnetic force, defined in a similar manner to the electric force, with the aid of the concept of a magnetic pole, but now *without* the possibility of a local contribution due to surrounding magnetic polarity if the point is inside the matter.

It is perhaps as well to emphasise the fact that both the electric and magnetic forces are defined in a theoretical manner which almost excludes the possibility of direct experimental verification. Electromagnetic measurements, particularly as regards the fields inside the matter, are concerned almost entirely with matter in bulk, and it is then only the mechanical or molar parts of these forces that are then under observation, the local parts, if they exist at all, being balanced on the spot by other forces of an origin not at present under review. We have however evidence that these local parts of the forces do exist.

(*d*) B, the magnetic induction, which is defined in the elementary theory in terms of the magnetic force H and the magnetic polarisation intensity I by the relation

$$B = H + 4\pi I,$$

and which is always assumed to be subject to the relation

$$\text{div } B = 0.$$

* *Cf.* my 'Theory of Electricity,' p. 363.

In the dynamical theory, however, this vector B appears as the complete æthereal magnetic force,* the part H being merely the mechanically effective part of this force in the material media. This new conception of the magnetic force, which is supported by such phenomena as the Hall effect, &c., where the deviations in ferro-magnetic media are proportional to the induction,† is not really inconsistent with the previous definition given under (*c*) above. As there explained, although the vector H is defined as the force on unit magnetic pole, there is still the possibility that there may exist in addition a force (4πI) not directly detectable in a mechanical experiment.

(*e*) In addition to these fundamental field vectors there are certain others of a subsidiary nature determining the electric and magnetic conditions in the media. One of these, the magnetic polarisation intensity I, has already been introduced, and there is an electric analogue in P, the dielectric polarisation intensity; these two vectors define respectively the effective resultant magnetic and electric bi-polar moments per unit volume of the media.

3. MAXWELL'S theory may now be expressed in the statement that in the most general case of interaction between the magnetic and electric fields the four fundamental vectors defining these fields are subject to the two differential vector equations

$$\text{Curl H} = \frac{4\pi\text{C}}{c},$$

$$\text{Curl E} = -\frac{1}{c}\frac{d\text{B}}{dt},$$

together with the scalar relation defining the electric charge density ρ

$$\text{div E} = 4\pi\rho.$$

The second vectorial equation practically implies that

$$\text{div B} = 0,$$

so that it is hardly necessary to add this as an additional equation.

For the further development of the theory all that is now necessary‡ is the kinematical specification of the current C in terms of the material vectors and an

* This was first clearly recognised by LARMOR. *Cf.* his papers "On a Dynamical Theory of the Electric and Luminiferous Medium," 'Phil. Trans.,' 1894–1897, or 'Æther and Matter' (Cambridge, 1900).

† The results are interpreted usually as implying that the deviations are proportional to the polarisation intensity in such cases, but this is equivalent to the statement given.

‡ We are not here concerned with the constitutional equations giving the laws of induction and conduction. These form a separate branch of the subject, whose results are irrelevant to the present discussion.

expression for the rate of change of the magnetic force vector at any point of the field. The first of these has the usual form

$$C = \frac{dD}{dt} + \rho\nu_m + C_1,$$

wherein C_1 is the density of the true conduction current, ν_m is the velocity of the material medium at the typical field point, and

$$D = \frac{1}{4\pi} E + P$$

is the total dielectric displacement of MAXWELL'S theory which consists in part of the dielectric polarisation P and in part of an æthereal constituent proportional to the electric force.

The time rate of change of the composite vector D requires careful specification; it consists in the main of the terms

$$\frac{1}{4\pi}\frac{dE}{dt} + \frac{dP}{dt},$$

but when the dielectric media are in motion there is in addition a term arising on account of the convection of the polarisation. This term has been shown* to be equal to

$$\text{Curl}\,[P\nu_m],$$

so that

$$\frac{dD}{dt} = \frac{1}{4\pi}\frac{dE}{dt} + \frac{dP}{dt} + \text{Curl}\,[P\nu_m],$$

The equation expressing the rate of change of the magnetic force is analogously

$$\frac{dH}{dt} = \frac{dB}{dt} - 4\pi\frac{dI}{dt} - 4\pi\,\text{Curl}\,[I\nu_m].$$

This latter equation must be specially emphasised as it has apparently never yet been explicitly introduced in the theory, although it is necessary to secure greater consistency in the dynamical theory. The expression $\frac{dI}{dt}$ represents the rate of change of the magnetic polarisation at a fixed point in the field only when the magnetic media as a whole are at rest. When these media are in motion there will be a contribution to this rate due to convection just as in the electric case, and the argument for its exact form may be developed on the same lines. The vectors B and H are, so to speak, attached to the æther, just as were the vectors D and E,† whilst

* *Cf.* my 'Theory of Electricity,' pp. 365–367, or LARMOR, 'Æther and Matter,' Chap. IV.

† The vector H being the composite vector of the magnetic theory is analogous to the vector D of the electric theory; the æthereal vector B is analogous to the æthereal vector E. This is the reverse of the usual convention, but see below § 10.

the vector I, as the vector P, is attached to the matter and moves with it. The last two equations contain, therefore, an explicit expression of the effect of the motion so that they are in a sense more convenient than the equations

$$D = \frac{1}{4\pi} E + P, \qquad H = B - 4\pi I,$$

defining merely the values of D and H at any point: they are, of course, ultimately consistent with these equations for, taking the second one as an example, we have

$$\frac{d}{dt} \operatorname{div} H = \frac{d}{dt} \operatorname{div} B - 4\pi \frac{d}{dt} \operatorname{div} - 4\pi \operatorname{div} \operatorname{Curl} [I\nu_m],$$

or

$$\frac{d}{dt} \operatorname{div} (H - B + 4\pi I) = 0.$$

With the possible exception of the equation defining dH/dt it is now generally agreed that the scheme here presented provides a completely effective specification of the kinematical connexions in the electromagnetic field.

To obtain some idea of the effect of these connexions on the dynamical processes operative in the field a further assumption is necessary, and this may take one of several forms which will be reviewed in the sequel. For the present we are concerned merely with these equations as effective representatives of the electromagnetic processes. They are sometimes given another form, by the introduction of a scalar potential ϕ and a vector potential A, these being such that

$$B = \operatorname{Curl} A, \qquad E = -\frac{1}{c} \frac{dA}{dt} - \operatorname{grad} \phi,$$

with the other two equations

$$\operatorname{Curl} H = \frac{4\pi}{c} C, \qquad \operatorname{div} E = 4\pi\rho.$$

The first two of these equations are equivalent to the remaining fundamental equation of MAXWELL'S theory which they replace, but they suffer from the serious disadvantage that the quantities A and ϕ specified in them are not completely defined by the equations as given and require additional data to fix them.

4. We have just noticed that an additional assumption of a dynamical character is necessary to render the Maxwellian electromagnetic scheme completely effective as an electrodynamic theory. The simplest and most direct form of this assumption may be taken to be that expressing the force of electrodynamic origin acting on an arbitrarily moving element of charge, this force being, per unit charge equal to

$$F = E + \frac{1}{c} [\nu B]$$

ν being the velocity of the charge and square brackets being employed as throughout to denote the vector product. This expression, first given in its complete form by LORENTZ and LARMOR, is generally regarded as being exact.*

But there are other forms of the additional assumption which are equally effective and in some respects more general than that just given. We may, for instance, assume definite expressions for the potential and kinetic energies of electromagnetic origin in the system and combine these with the assumption that the dynamical processes operative in the field are governed by the same general laws as are the processes in a similar mechanical system. This form of the argument proves ultimately to be consistent with the first as regards the expression for the effective mechanical force on a moving element of charge, but it has the advantage of being expressed in more general terms, thus carrying with it the possibility of fitting better with any modification that it may subsequently be thought desirable to make in our general conception of the theory. It need not be presumed that this form of the argument is any less general than the first on account of the fact that it apparently involves more than one assumption, for this increase in number is counterbalanced by the fact that the dynamical argument ultimately reduces the two fundamental equations to one, FARADAY'S relation being derived simultaneously as a consequence of AMPÈRE'S.

The general dynamical argument was first formulated by MAXWELL† for the case of the field surrounding a system of linear currents. His analysis was subsequently extended to cover the more general case, firstly by VON HELMHOLTZ‡ and LORENTZ,§ later by LARMOR,|| MACDONALD,¶ ABRAHAM,** and others. In the later investigations the whole subject is regarded from the point of view of the theory of electrons, wherein every manifestation of the field is regarded as arising in the aggregate disposition or motion of electronic charges; even in the former investigations, although they are apparently of a more general character, certain assumptions are involved which render their analyses theoretically effective only under the same restricted circumstances. In each of these cases it is assumed, practically speaking, that the potential energy of the electromagnetic field is represented by the expression

$$W = \frac{1}{8\pi}\int E^2\,dv$$

and the kinetic energy by

$$T = \frac{1}{2c}\int (AC)\,dv,$$

* Relatively modifications are not here under consideration.

† 'Treatise,' vol. 2, Ch. VI., VII.

‡ 'Ann. Phys. Chem.,' vol. 47 (1892), p. 1.

§ 'La Théorie Électromagnétique de MAXWELL' (Leiden, 1892), §§ 55–61.

|| 'Æther and Matter,' § 50.

¶ 'Electric Waves,' Appendix I.

** 'Ann. Phys.,' vol. 10 (1903), p. 105.

both volume integrals being taken throughout the entire field: the derivation of the dynamical and field equations is then accomplished by an application of one or the other of the well-known processes of analytical dynamics. The interpretation of the same results for the case when the kinetic energy is given by the usual expression

$$T = \frac{1}{8\pi}\int B^2\,dv$$

was given by the present author.*

Whilst the theoretical simplicity of these discussions, which results from their interpretation in terms of the simple electronic hypothesis, is a great point in their favour, it seemed, nevertheless, of theoretical interest at least to attempt to formulate the problem under less restricted conditions, especially in view of the pronounced tendency exhibited in some quarters to deny the adequacy of the Maxwellian theory as a complete microscopic theory. Besides the more general discussion in the form in which it is here presented emphasises certain difficulties inherent in the usual formulations which have not hitherto received adequate attention.

5. The most general dynamical principle which determines the motion of every material system is the Law of Least Action expressible in the usual form

$$\delta\int_{t_1}^{t_2} L\,dt \equiv \delta\int_{t_1}^{t_2}(T-W)\,dt = 0$$

wherein T denotes the kinetic energy and W the potential energy of the system in any configuration and formulated in terms of any co-ordinates that are sufficient to specify the configuration in accordance with its known properties and connexions, and where the variation refers to a fixed time of passage of the system from the initial to the final configuration. This is the ordinary form of HAMILTON's principle, but it involves in any case a complete knowledge of the constitution of the systems, because, before it can be applied it is necessary to know the exact values of the kinetic and potential energies expressed properly in terms of the co-ordinates and velocities. As however we have frequently to deal with systems whose ultimate constitution is either wholly or partly unknown it is necessary to establish a modified form of the principle allowing for a possible ignorance of the constitution of the systems with which we may have to deal. The modification is fully discussed in most works on analytical dynamics,† and we may here content ourselves by merely presenting the results, interpreting them however in a manner somewhat different from that usually given, in order to throw some light on certain questions which arise in the subsequent application in our present theory. Suppose then that it has been found impracticable to express the Lagrangian function L in terms of the chosen co-ordinates of the system, the typical one of which we may denote by q; but that it is expressed in

* 'Phil. Mag.,' vol. 32 (1916), p. 195.

† *E.g.*, 'Treatise on Analytical Dynamics' (2nd ed., Cambridge, 1918), by E. T. WHITTAKER.

terms of a certain number of variables $x_1, x_2, \ldots x_k$, which are known to be connected with the co-ordinates q and their velocities $\dot{q}$ by a series of relations of the type

$$M_s = 0$$

M_s being a function of the co-ordinates q, the velocities $\dot{q}$, the variables x and the differential coefficients of these latter variables with respect to the time. For the sake of simplicity we shall restrict our statement to the case when the first differentials only appear. The usual method of procedure is to introduce a set of multipliers λ_s, functions of the time, and then to consider the variations of the integral

$$\int_{t_1}^{t_2} (L + \Sigma \lambda_s M_s)\, dt$$

where the q's and x's undergo independent variations. The equations obtained for the vanishing of the variation are of two types. Firstly, there is an equation of the type

$$\frac{d}{dt}\left(\Sigma_s \lambda_s \frac{\partial M_s}{\partial \dot{x}}\right) - \Sigma \lambda_s \frac{\partial M_s}{\partial x_s} - \frac{\partial L}{\partial x} = 0$$

for each variable x: these with the restricting equations will determine the x's and λ's as functions of the co-ordinates q and the time. Then there is an equation of the type

$$\frac{d}{dt}\left(\Sigma_s \lambda_s \frac{\partial M_s}{\partial \dot{q}}\right) - \Sigma \lambda_s \frac{\partial M_s}{\partial q} = 0$$

for the motion in each q co-ordinate.

The latter equations only involve the Lagrangian function L through the quantities λ and x which enter into it, and once these are determined the rest of the solution involves only the restricting conditions. In fact when once these multipliers and variables are determined and regarded as functions of the time only the motion in the q co-ordinates is completely determined by the condition that the integral

$$\int \Sigma \lambda_s M_s\, dt$$

is stationary for independent variations of the co-ordinates q. It may even happen that the relations M involve the co-ordinates q and the variables x in such a way that it is possible to separate M into two terms, one of which is a function explicitly of the q's only and the other of the x's only. In this case the part of the integral required in the above statement is only that part of it involving the q's and this is independent entirely of the co-ordinates x.

This remark has an important bearing on a question which occurs in the sequel, and it shows that the existence of a variational form for the equations of motion does not necessarily imply that the integrand involved is a true Lagrangian function for the system.

6. Now let us apply these principles to our electromagnetic problem. The conditions in the field surrounding a number of bodies are specified in the usual way by the magnetic induction vector B and the electric force vector E, and the part of the Lagrangian function associated with this field may be taken to be

$$\frac{1}{8\pi}\int (B^2-E^2)\,dv,$$

the first term denoting the magnetic or kinetic energy and the second the electric or potential, and the integral is extended over the whole of space. In addition to these energies there will be the energies of the material bodies in the field which will consist in part of the kinetic energies of their organised motions, in part of their potential energy relative to one another or to any extraneous fields of non-electric nature, and in part finally of internal energy of elastic or motional type in the media. The part of the Lagrangian function corresponding to these energies can, in the most general case, be denoted by

$$\int (L_0-W_i)\,dv$$

where L_0 is the Lagrangian function of the organised motions of the media, reckoned per unit volume at each place and assumed to be a function only of the position co-ordinates and velocities, and W_i is the internal energy of all types reckoned as potential energy per unit volume: this latter term will be a function of the electric and magnetic polarisations in the media, but will be assumed not to depend to any appreciable extent on the rates of variation of these conditions, and in so far as some of the internal energy is essentially of kinetic type, it will be in reality a sort of modified Lagrangian function with the energy corresponding to the motional terms converted to potential energy in the usual way. The function L_0 may also be taken to include a part arising from the assumed inertia of any free electrons that may be present.

The motion of the system can now be expressed in the form

$$\delta\int_{t_1}^{t_2} dt \int \left[L_0-W_i+\frac{1}{8\pi}(B^2-E^2)\right] dv$$

and we could conduct the variation directly were it not for the fact that our functions are not all expressed explicitly in terms of the independent co-ordinates of the systems, which are in reality the position co-ordinates of the elements of matter and electricity. As indicated above we can however avoid the use of any such explicit interpretation by the use of undetermined multipliers. In this way the variations of E and B can be temporarily rendered independent of each other and of the actual co-ordinates of the material and electrical elements.

We know that the vectors of the theory are connected with one another and the actual co-ordinates of the system by the equations

$$\operatorname{div} \mathrm{E} = 4\pi\Sigma e - 4\pi \operatorname{div} \mathrm{P}$$

$$\operatorname{Curl} \mathrm{H} = \frac{1}{c}\frac{d\mathrm{E}}{dt} + \frac{4\pi}{c}\frac{d\mathrm{P}}{dt} + \frac{4\pi}{c}\operatorname{Curl}[\mathrm{P}\dot{r}_m] + \frac{4\pi}{c}\Sigma e\,(\dot{r}_m + \dot{r}_e)$$

$$\frac{d\mathrm{B}}{dt} = \frac{d\mathrm{H}}{dt} + 4\pi\frac{d\mathrm{I}}{dt} + 4\pi \operatorname{Curl}[\mathrm{I}\dot{r}_m].$$

In these equations P is the dielectric and I the magnetic polarisation intensity; $\dot{r}_m$ is the velocity and r_m the position vector of the element of matter and $\dot{r}_e$ and r_e the velocity and position vectors relative to this element of the typical element of free charge (e) over which the sum Σ in the first and second equations is taken *per unit volume* at each place.

In these equations we have purposely refrained from assuming a definite electronic constitution for the dielectric and magnetic polarisations as it was desired to emphasise certain points in connexion with the mechanical forcive which have not yet been adequately dealt with.

We have thus to introduce three undetermined multipliers one scalar ϕ and two vectors A_1, A_2 and it is then the variation of

$$\int_{t_1}^{t_2} dt \int dv \Big[\mathrm{L}_0 - \mathrm{W}_i + \frac{1}{8\pi}(\mathrm{B}^2 - \mathrm{E}^2) + \frac{\phi}{4\pi}(\operatorname{div}\mathrm{E} + 4\pi \operatorname{div}\mathrm{P})$$
$$- \frac{1}{4\pi}\Big(\mathrm{A}_1, \operatorname{Curl}\mathrm{H} - \frac{1}{c}\frac{d\mathrm{E}}{dt} - \frac{4\pi}{c}\frac{d\mathrm{P}}{dt} - \frac{4\pi}{c}\operatorname{Curl}[\mathrm{P}\dot{r}_m]\Big)$$
$$+ \frac{1}{4\pi c}\Big(\mathrm{A}_2, \frac{d\mathrm{B}}{dt} - \frac{d\mathrm{H}}{dt} - 4\pi\frac{d\mathrm{I}}{dt} - 4\pi\operatorname{Curl}[\mathrm{I}\dot{r}_m]\Big)$$
$$- \Sigma e\phi + \frac{1}{c}\Sigma e\,(\mathrm{A}_1, \dot{r}_m + \dot{r}_e)\Big]$$

that is to be made null, afterwards determining the forms of the various undetermined functions to satisfy the restrictions which necessitated their introduction. In conducting the variation we can now treat the electric force, displacement and polarisation, the magnetic force induction and polarisation and the position co-ordinates of the electrical and material elements as all independent. We here see the reason for introducing the equation expressing the rate of change of H instead of the equation

$$\mathrm{H} = \mathrm{B} - 4\pi\mathrm{I}$$

determining its value, for this latter equation does not in reality enable us to obtain a relation between the variations of H and the position co-ordinates of the matter, so we could not treat all our variables as independent.

In the main the details of the variational calculation possess no novel features and need not here be elaborated. There are, however, one or two terms which require careful handling, especially when finding the variations due to the alteration of position of the matter.

The variations of terms of the type

$$\int \phi \operatorname{div} \mathrm{P}\, dv, \quad \frac{1}{c}\int\left(\mathrm{A}_1 \frac{d\mathrm{P}}{dt}\right) dv, \quad \frac{1}{c}\int(\mathrm{A}_1 \operatorname{Curl}[\mathrm{P}\dot{r}_m])\, dv,$$

due to variations in the position co-ordinates of the elements of matter to which the vectors P and r_m are attached should not be performed until the differential operators affecting the P function are eliminated by an integration by parts. If we bear this in mind we shall find that the final result for the variation consists of terms at the time and space limits, which require separate adjustment, together with

$$\begin{aligned}\int_{t_1}^{t_2} dt \int dv \Bigg[\delta \mathrm{L}_0 &+ \frac{1}{4\pi}\left(\delta\mathrm{B},\ \mathrm{B} - \frac{1}{c}\frac{d\mathrm{A}_2}{dt}\right) - \frac{1}{4\pi}\left(\delta\mathrm{E},\ \mathrm{E} + \nabla\phi + \frac{1}{c}\frac{d\mathrm{A}_1}{dt}\right) \\
&- \frac{1}{4\pi}\left(\delta\mathrm{H},\ \operatorname{Curl}\ \mathrm{A}_1 - \frac{1}{c}\frac{d\mathrm{A}_2}{dt}\right) - \left(\mathrm{P},\ (\delta r_m \nabla)\left(\nabla\phi + \frac{1}{c}\frac{d\mathrm{A}_1}{dt}\right)\right) \\
&- \frac{1}{c}\left(\delta r_m,\ \left[\operatorname{Curl}\ \mathrm{A}_1,\ \frac{d\mathrm{P}}{dt}\right]\right) - \frac{1}{c}\left(\delta r_m \left[\operatorname{Curl}\ \frac{d\mathrm{A}_1}{dt},\ \mathrm{P}\right]\right) \\
&- \frac{1}{c}(\delta r_m \nabla)(\operatorname{Curl}\ \mathrm{A}_1,\ [\mathrm{P}\dot{r}_m]) + \frac{1}{c}\left(\mathrm{I},\ (\delta r_m \nabla)\frac{d\mathrm{A}_2}{dt}\right) \\
&+ \frac{1}{c}\left(\delta r_m,\ \left[\operatorname{Curl}\ \mathrm{A}_2,\ \frac{d\mathrm{I}}{dt}\right]\right) + \frac{1}{c}\left(\delta r_m \left[\operatorname{Curl}\ \frac{d\mathrm{A}_2}{dt},\ \mathrm{I}\right]\right) \\
&+ \frac{1}{c}(\delta r_m \nabla)(\operatorname{Curl}\ \mathrm{A}_2,\ [\mathrm{I}\dot{r}_m]) \\
&+ \Sigma e\left(\delta r_m + \delta r_e,\ -\nabla\phi - \frac{1}{c}\frac{d\mathrm{A}_1}{dt} + \frac{1}{c}[\dot{r}_m + \dot{r}_e,\ \operatorname{Curl}\ \mathrm{A}_1]\right) \\
+ \delta\mathrm{W}_i &+ \left(-\nabla\phi - \frac{1}{c}\frac{d\mathrm{A}_1}{dt} + \frac{1}{c}[\dot{r}_m,\ \operatorname{Curl}\ \mathrm{A}_1],\ \delta\mathrm{P}\right) \\
&+ \left(\frac{1}{c}\frac{d\mathrm{A}_2}{dt} - \frac{1}{c}[\dot{r}_m,\ \operatorname{Curl}\ \mathrm{A}_2],\ \delta\mathrm{I}\right)\Bigg],\end{aligned}$$

where in the terms $(\delta r_m \nabla)$ (Curl A_1, $[\mathrm{P}\dot{r}_m]$) and $(\delta r_m \nabla)$ (Curl A_2, $[\mathrm{I}\dot{r}_m]$) the vector operator ∇ $\left(\text{whose components are } \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ is presumed to affect only the functions A_1 and A_2.

The variations δr_m, δr_e which determine the virtual displacements of the electrical and material elements and the variations δE, δB, δP, ..., can now be considered as all independent and perfectly arbitrary, and hence the coefficient of each must vanish separately in the dynamical variational equation. The coefficients of the latter variations lead to the equations

$$B = \frac{1}{c}\frac{dA_2}{dt}, \qquad E = -\nabla\phi - \frac{1}{c}\frac{dA_1}{dt},$$

$$\text{Curl } A_1 = \frac{1}{c}\frac{dA_2}{dt},$$

from which it follows that the multipliers ϕ and A_1 are the ordinary scalar and vector potentials of the theory so that further

$$\text{Curl } E = -\frac{1}{c}\frac{d}{dt}\text{Curl } A_1.$$

As regards the vector A_2, this is a new vector potential whose *curl* is required in our subsequent discussions. For this we have

$$\text{Curl } B = \text{Curl } H + 4\pi \text{ Curl } I = \frac{1}{c}\frac{d}{dt}\text{Curl } A_2,$$

whilst if we use C_e as the total current of true electric flux we have, by AMPÈRE's equation

$$\text{Curl } H = \frac{1}{c}\frac{dE}{dt} + \frac{4\pi}{c}C_e.$$

Thus, if we use

$$C'_e = C_e + c \text{ Curl } I,$$

we have

$$\frac{d}{dt}(\text{Curl } A_2) = \frac{dE}{dt} + 4\pi C'_e.$$

The main part of *Curl* A_2 is therefore represented by the electric force: there is however in addition a local term E_0 depending on the time integral of the current density at the point. We can thus write

$$\text{Curl } A_2 = E + E_0.$$

If we use the values thus determined for the various undetermined multipliers introduced at the outset, the remaining terms of the variation give for the motion of the material and electrical elements equations of the type

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L_0}{\partial \dot{x}_m}\right) - \frac{\partial L}{\partial x_m} = \left(P\frac{\partial E}{\partial x_m}\right) + \left(I\frac{\partial B}{\partial x_m}\right) + \frac{1}{c}\left([P\dot{r}_m], \frac{\partial B}{\partial x_m}\right) \\ - \frac{1}{c}\left([I\dot{r}_m], \frac{\partial (E+E_0)}{\partial x_m}\right) - [P, \text{ Curl } E]_x \\ - [I, \text{ Curl } B]_x - \frac{1}{c}\left[B\frac{dP}{dt}\right]_x + \frac{1}{c}\left[E+E_0, \frac{dI}{dt}\right]_x \\ + \rho\left(E_x + \frac{1}{c}[\dot{r}_m B]_x\right) + \frac{1}{c}[C_1 B]_x,\end{aligned}$$

for the motion of the matter and of type

$$\frac{d}{dt}\left(\frac{\partial \mathrm{L}_0}{\partial \dot{x}_m}\right) - \frac{\partial \mathrm{L}}{\partial x_m} = e\left(\mathrm{E}_x + \frac{1}{c}[\nu \mathrm{B}]_x\right)$$

for the electrical elements. In the first of these equations $\rho = \Sigma e$ is the density of the free charge, and $\mathrm{C}_1 = \Sigma e\dot{r}_e$ is the density of the true conduction current; in the second equation $r = r_m + r_e$ is the absolute position co-ordinate of the electrical element and $\nu = \dot{r}$ is its resultant velocity.

In addition we have the variational equation for the internal energy which can be left as it stands in the form

$$\delta \mathrm{W}_i = \left(\mathrm{E} + \frac{1}{c}[\dot{r}_m \mathrm{B}],\ \delta \mathrm{P}\right) + \left(\mathrm{B} - \frac{1}{c}[\dot{r}_m,\ \mathrm{E}+\mathrm{E}_0],\ \delta \mathrm{I}\right).$$

Interpreted in terms of the language of ordinary dynamics these equations imply that

$$(\mathrm{P}\nabla)\,\mathrm{E}_x + (\mathrm{I}\nabla)\,\mathrm{B}_x + \frac{1}{c}\left([\mathrm{P}\dot{r}_m],\ \frac{\partial \mathrm{B}}{\partial x}\right) - \frac{1}{c}\left([\mathrm{I}\dot{r}_m],\ \frac{\partial(\mathrm{E}+\mathrm{E}_0)}{\partial x}\right)$$
$$+\frac{1}{c}\left[\frac{d\mathrm{P}}{dt} + \mathrm{C}_1,\ \mathrm{B}\right]_x - \frac{1}{c}\left[\frac{d\mathrm{I}}{dt},\ \mathrm{E}+\mathrm{E}_0\right]_x + \rho\left(\mathrm{E}_x + \frac{1}{c}[\dot{r}_m \mathrm{B}]_x\right)$$

is the force per unit volume at each place tending to accelerate the motion of the matter, whilst

$$e\left(\mathrm{E} + \frac{1}{c}[\nu \mathrm{B}]\right)$$

is the force tending to accelerate the motion of the element of electricity e.

The electrical terms in the first of these results are consistent with those obtained by LARMOR* and others, but the magnetic terms, which are in fact analogous with the electrical terms depending on the dielectric polarisation, are fundamentally different from those obtained by these authors. Further, since

$$\mathrm{E} = -\frac{1}{c}\frac{d\mathrm{A}_1}{dt} - \nabla\phi,$$

it follows that

$$\text{Curl } \mathrm{E} = -\frac{1}{c}\frac{d}{dt}\text{Curl } \mathrm{A}_1$$
$$= -\frac{1}{c}\frac{d\mathrm{B}}{dt},$$

so that our equations are in complete agreement with the most general form of MAXWELL'S theory.

* 'Æther and Matter,' Chap. VI.

7. If we examine the above analyses closely we shall notice a rather important point bearing on a fundamental question which has already been the subject of some discussion.* If we take the integral in its complete form with the variation carried out and with the values of the various multipliers inserted it can be seen to reduce to the expression

$$\int_{t_1}^{t_2} dt \int dv \Big[\delta L_0 - \delta W_i + (P\,\delta E) + (I\,\delta B) - \frac{1}{c}((\delta r_m \nabla),\ B,\ [P\dot{r}_m]) - (\delta r_m,\ [P \text{ Curl } E])$$
$$- \frac{1}{c}((\delta r_m \nabla)(E+E_0),\ [I\dot{r}_m]) - (\delta r_m [I \text{ Curl } B])$$
$$+ \left(\frac{\delta r_m}{c},\ \left[\frac{dP}{dt},\ B\right]\right) - \frac{1}{c}\left(\delta r_m,\ \left[\frac{dI}{dt},\ E+E_0\right]\right)$$
$$+ \left(E + \frac{1}{c}[\dot{r}_m B],\ \delta P\right) + \left(B - \frac{1}{c}[\dot{r}_m,\ E+E_0],\ \delta I\right)$$
$$+ \Sigma e\left(\delta r_m + \delta r_e,\ E + \frac{1}{c}[VB]\right)\Big],$$

in which neither the electric nor magnetic energy contributes an explicit term. This is the first definite indication we have that the modified function with which we may operate to find the equations of motion of the electric and material elements is explicitly independent of the expression for the energy in the æthereal field. We may, in fact, see that, just as in the dynamical problem examined above, the whole circumstances of the motion in the real co-ordinates of the system can be derived by the variational principle, using the integral

$$\int_{t_1}^{t_2} dt \int dv \Big[L_0 - W_i - \phi \text{ div } P - \frac{1}{c}\left(A_1\,\frac{dP}{dt}\right) - \frac{1}{c}(A_1 \text{ Curl } [P\dot{r}_m])$$
$$- \frac{1}{c}\left(A_2\,\frac{dI}{dt}\right) - \frac{1}{c}(A_2 \text{ Curl } [I\dot{r}_m]) + \Sigma e\phi + \Sigma e\,(A_1,\ \dot{r}_m + \dot{r}_e)\Big]$$

just as we used the Hamiltonian integral, taking in it E, A_1, A_2, ϕ as functions of the time and space co-ordinates only. It is of course possible to establish this directly, for it is easily verified that the difference between the integrand just employed and the previous one, viz.,

$$L_0 - W_i + \frac{1}{8\pi}(B^2 - E^2),$$

involves only complete differentials with respect to the time or space co-ordinates. This difference therefore integrates out to the limits and remains ineffective as regards the general dynamical variational equations, and we can therefore use either integrand indiscriminately.

* *Cf.* 'Phil. Mag.,' vol. 32 (1916), p. 195, where references to previous work are given.

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This is the general form of the result obtained by SCHWARZSCHILD* that in the special case when we are concerned only with free electrons moving in an æthereal field free from matter their equations of motion can be derived by the variational principle, using the integral

$$\int_{t_1}^{t_2} dt \left[L_0 - \Sigma e\phi + \Sigma \frac{1}{c} (A\dot{r}) \right],$$

where ϕ, A, the ordinary scalar and vector potentials, are regarded as functions of the time and space variables only.

We now see why it is that consistent formulæ have been obtained by different authors using apparently different expressions to represent the field energies. The results are in fact all explicitly independent of any particular interpretations for these energy expressions.

8. The general dynamical formulation of § 6 agrees with the fact that the material media of the field have an internal constitution which enables them to resist the setting up of the electric and magnetic polarisations by forces $E + \frac{1}{c}[\dot{r}_m B]$ and $B - \frac{1}{c}[\dot{r}_m, E + E_0]$ respectively, and that in consequence of any change in the polarised state of these media their intrinsic energy of elastic or motional type is increased by the amount

$$\int dv \int_1^2 (E' \delta P) + (B' \delta I),$$

where we have used

$$E' = E + \frac{1}{c}[\dot{r}_m B], \qquad B' = B - \frac{1}{c}[\dot{r}_m, E + E_0].$$

Thus in setting up the electric field and its associated dielectric polarisations in the medium the potential energy of the field is increased by the amount

$$\frac{1}{8\pi} \int E^2 \, dv$$

on account of the establishment of the æthereal field together with

$$\int dv \int (E' \delta P)$$

on account of the material polarisation, both amounts being reckoned as potential energy.

This gives a total for the field equal to

$$\int dv \int \left\{ \frac{1}{4\pi} (E \delta E) + (E \delta P) + \frac{1}{c} ([\dot{r}_m B] \delta P) \right\}.$$

* 'Gött. Nachr. (Math.-phys. Kl.),' 1903, p. 125.

This result is consistent with that generally obtained in these theories. The last term arising on account of the current due to the convection of the polarisations is however probably of kinetic origin.

Of course, in the general case, all the potential energy put into the field cannot be got out of it again in the form of useful mechanical work, or, in other words, it is not all available. The effectively available energy in the present case consists in the main of the part

$$\int dv \left(\frac{E^2}{8\pi} + \int (P\,\delta E)\right).$$

For the magnetic polarisations the results are somewhat different. In this case the kinetic energy of the field is assumed to be

$$\frac{1}{8\pi}\int B^2\,dv,$$

to which we must add the intrinsic kinetic energy involved in the induced magnetic polarisations to obtain the total energy in the field; reckoned as potential energy the intrinsic energy is

$$\int dv \int (B'\,dI),$$

and therefore as kinetic energy it is

$$-\int dv \int (B'\,dI),$$

giving a total for the field equal to

$$\frac{1}{8\pi}\int B^2\,dv - \int dv\,(B'\,dI)$$

$$= \frac{1}{4\pi}\int dv \int (B\,dH) + \frac{1}{c}\int dv \int (E+E_0,\ [dI,\ \dot{r}_m]),$$

a result which is again practically equivalent to that usually given in this theory.

If we treat the convection of the dielectric polarisation as effectively equivalent to a magnetic polarisation of intensity

$$I' = \frac{1}{c}[P\dot{r}_m],$$

and the convection of the magnetic polarisation as effectively equivalent to a dielectric polarisation of intensity

$$P' = -\frac{1}{c}[I\dot{r}_m],$$

and include the energies corresponding to these two parts in with the appropriate totals the formulæ just obtained are somewhat simpler in form. The total kinetic energy will be now

$$\frac{1}{8\pi}\int\left[B^2-8\pi\int(B\,dI)-8\pi\int(B\,dI')\right]dv$$

or

$$\int\frac{dv}{4\pi}\int(B\,dH'),$$

where

$$B = H'+4\pi I+\frac{4\pi}{c}[P\dot{r}_m],$$

and the potential energy is

$$\frac{1}{8\pi}\int\left[E^2+8\pi\int(E\,\delta P)+8\pi\int(E\,\delta P')\right]dv,$$

or is

$$\iint(E\,dD')\,dv,$$

where

$$D' = \frac{E}{4\pi}+P+P'.$$

This, however, leaves out of account the term

$$\int dv\int(E_0\,\delta P'),$$

which, representing as it does the energy of local actions and reactions, may be assumed to be inoperative as regards general mechanical processes, and may in consequence be rejected altogether.

In connexion with the discussions of the expressions for the internal energy of the material media it may be worth while emphasising the significance of the various terms in the expression for the forcive on the polarised material elements derived in the previous paragraph, especially the way in which the various terms arise and their dependence on the term

$$\text{Curl}\,[P\nu_m]$$

in the complete current. Further, we may notice how the complete expression for the change of intrinsic energy per unit volume, viz.,

$$\left(E+\frac{1}{c}[\nu_m\,B],\,\delta P\right)$$

is derived, the latter term being due to the above-mentioned term in the current.

This last remark points to the possibility of obtaining an elementary deduction of the expression

$$E + \frac{1}{c}[\nu_m, B]$$

for the complete electromotive forcive simply by calculating the rate of change of intrinsic energy of a moving bi-pole, and the calculation has in fact been carried out by LARMOR,* taking however a parallel plate condenser with equal and oppositely charged plates, moving in a uniform magnetic field.

An analogous argument in the magnetic case will give a deduction of the magneto-motive forcive

$$B - \frac{1}{c}[\nu_m, E].$$

9. We have stated that the magnetic energy expressions just obtained are effectively equivalent to those usually derived, whereas as a matter of fact this is true only of the final result; the various formulæ employed in the derivation of this result are not in their usual form but it has been shown elsewhere† that they are consistent with the complete dynamical theory, the more usual formulæ and the various modifications of them which have from time to time been suggested being all inadmissible on this score. A complete discussion of the justification for this last statement is necessarily beyond the scope of the present paper, but it may perhaps serve a useful purpose if a brief outline of some of the more important reasons is given, especially as they have some bearing on points raised elsewhere in the present discussion.

In the first place there is probably little or no difficulty in seeing the fallacy in the usual and simplest form of the theory wherein the expression $\mu H^2/8\pi$ for the magnetic energy density is derived in the statical theory as potential energy and in the dynamical theory as kinetic energy: we need only enquire as to the type of energy represented by the same expression when the field is due in part to rigid magnets and in part to steady currents. The more consistent result is obtained by taking

$$\frac{\mu H^2}{8\pi} - \frac{1}{4\pi}(HB)$$

as the expression for the density in the statical case as this agrees with the opposite sign in the dynamical case and yet gives the same total.

There is however another form of the results first tentatively suggested by HERTZ and HEAVISIDE and subsequently developed in great detail by other writers, more particularly by R. GANS‡ and H. WEBER, wherein the difficulty presents itself in

* 'Proc. Lond. Math. Soc.' (1915).

† *Cf.* my 'Theory of Electricity,' p. 417, or 'Roy. Soc. Proc.,' vol. 93, A, p. 20 (1916).

‡ 'Ann. der Physik,' vol. 13 (1904), p. 634, and 'Encyklopädie der Math. Wissensch.,' vol. v., art. 15, where references to other authors will be found.

another and more involved form. The fundamental basis of this theory is the assumption of a distribution of true magnetic matter of density at any place equal to

$$\rho_m = -\text{div } I_0 = \frac{1}{4\pi} \text{div } B$$

wherein I_0 is the density of the permanent magnetic polarity. This magnetic matter is supposed to be distributed continuously throughout the space but so that the amount in any portion of the matter is zero, a condition which is perhaps rather difficult of realisation, as it would make the distribution in any particular portion of the matter dependent on the distribution in all the surrounding portions.

In this theory the magnetic energy is first calculated on analogy with the electrostatic energy; the magnetic induction vector B is regarded as a sort of composite displacement produced by the acting force H, so that the energy per unit volume is

$$\frac{1}{4\pi} \int^{B} (H\, dB).$$

This expression is then verified to be equivalent in the purely statical case to the volume integral

$$\int dv \int^{\rho_m} \phi \, d\rho_m$$

taken over the entire field, the surface integral over the infinite boundary contributing nothing in all regular cases; ϕ is the magnetic potential of the field.

In generalising the theory to the case where the field is due to linear currents the same physical basis is adopted as regards the expression

$$\int dv \int^{B} (H\, dB),$$

which still therefore remains valid, and when there are no permanent magnets about this is easily verified by the usual argument to be equivalent to the summation

$$\frac{1}{c} \Sigma \int^{N} J\, dN$$

over the different current elements, J denoting the typical current strength and N the induction through its circuit. When there are permanent magnets present this expression becomes

$$\int dv \int^{\rho_m} \phi \, d\rho_m + \frac{1}{c} \Sigma \int^{N} J\, dN.$$

It is then shown that the mechanical forcive on the magnetic matter in any one of its co-ordinates is derivable as the appropriate *negative* gradient of this energy function, whilst the force on a current is to be obtained as the *positive* gradient with respect to its position co-ordinate.

Unfortunately all the authors concerned merely talk of magnetic energy without specifying whether it is to be taken as kinetic or potential energy. One might perhaps infer that as the results are interpreted in terms of a static potential function it is implied that all the energies are potential, but the fact that the forces on the currents are derivable as the positive gradients of the function

$$\frac{1}{c}\Sigma\int^{\mathrm{N}}\mathrm{J}\,d\mathrm{N}$$

suggests that this part of the energy at least is kinetic energy. The difficulty of sign is therefore still present.

Even if we confine ourselves to the statical theory the same interpretation is not entirely free from difficulties of another kind. The potential energy in the field is taken to be represented by

$$\int dv\int^{\rho_m}\phi\,d\rho_m,$$

but this expression really represents the total energy in the field; in the general case the only part of this energy which is mechanically available is

$$\int dv\int^{\phi}\rho_m\,d\phi,$$

and this is properly speaking the potential function from which the mechanical forces acting on the magnetism are to be derived. Of course, when the law of induction is linear the intrinsic energy of the field is equal to the available energy, but even then their natures are fundamentally different and equality in their magnitudes is hardly a sufficient justification for confusing the one with the other.

Apart from this difficulty, however, the next step employed in the development of the theory will cause some trouble. To effect the transformation from the expression

$$\int dv\int^{\rho_m}\phi\,d\rho_m$$

to the equivalent expression

$$-\int dv\int^{\mathrm{I}_0}(\mathrm{H}\,d\mathrm{I}_0)$$

the method of integration by parts is employed. But LARMOR has shown that two expressions of this type being derived the one from the other by the method of integration by parts, really represent fundamentally different distributions of the energy in the field, although the total amounts represented by them are the same. The two expressions cannot therefore be used indiscriminately to determine the stresses around an element of the magnetic matter. It is not, of course, possible at the present stage to say which of the two expressions does represent the true energy distribution, but any examination of the mechanical inter-relations of the different magnetic masses in the field postulates a previous decision as to the proper expression to be taken as representing the available energy of these masses in its normal form, respecting both its total amount and proper distribution. Once this decision has been made it is unsafe to employ the method of integration by parts unless due regard is paid to the surface integrals thereby introduced.

The distribution of energy interpreted in terms of ideal magnetic matter which is properly equivalent to the expression

$$-\int dv \int^{I_0} (H\, dI_0)$$

is such that the energy in any volume of the field consists of a distribution throughout it of density at each point equal to

$$-\int \phi \operatorname{div} dI_0 = -\int \phi \, d\rho_m$$

together with a surface distribution of density

$$\int \phi \, dI_n$$

over its surface. This corresponds properly to POISSON's distribution of magnetic matter and emphasises the necessity for the inclusion of surface distributions of magnetic matter.

This explains why it is that the above theory determines properly the forces on the permanent magnets as a whole, but fails to give a consistent account of the internal forces between different parts of the same magnet. At the surface of an ordinary magnet it may quite legitimately be assumed that owing to the existence of a transition layer, the normal component of the magnetisation vanishes there, and consequently the surface integrals applied to the outer surfaces of any such body would also vanish; the two different expressions for the contained energy thus become equivalent.

This is perhaps sufficient to justify a summary rejection of this new interpretation of the energy relations of the magnetic field, as being at most no better than the older one which it presumed to displace. The real trouble in both cases seems to have arisen mainly in an effort to discover an analogy in the relations of the electric and magnetic fields. HERTZ and HEAVISIDE were the first to insist on the existence of this analogy, and practically all the modern writers follow them in this matter, even so far as to regard it as providing sufficient justification for certain fundamental formulæ of the theory. A close scrutiny of the subject will, however, reveal the fact that although the mathematical relations connecting the magnetic force induction and polarisation are to a certain extent similar to those connecting the electric force, total electric displacement and dielectric displacement, the similarity ends with these relations, and the dynamical characteristics of the two types of field are essentially different; and the analogy itself, so far as it exists, seems to be based on erroneous and confused conception of the nature of the magnetic energy as determined by the usual expression of the theory, so that it finds no place in a consistent formulation of the subject, notwithstanding even HEAVISIDE's spirited defence in criticism of LARMOR.

In his treatise ABRAHAM adopts the analogy as a sufficient basis for the discussion of the magnetic theory, but decides that the procedure is not without its difficulties, particularly as regards the ferro-magnetic phenomena; not being able to overcome these he condemns the whole procedure as being inadequate to include a proper account of these matters.

10. We now turn from these discussions to a brief review of the general energy relations of the electromagnetic field. A concise account of these relations so far, that is, as they have been dealt with in existing accounts of the subject, has been given with full references in my treatise (Ch. XIV.), and it will suffice for the present to give the barest outlines of the discussion so far as they may be required.

The fundamental equation expressing in its most concise form the energy principle for the electromagnetic field can be written in the form

$$\frac{dW}{dt}+\frac{dT}{dt}+F+\int S_n\,df = 0,$$

wherein W and T are respectively the potential and kinetic energies inside any volume bounded by the closed surface f in the field, F is the rate of dissipation of electromagnetic energy into energy of other types such, for instance, as results mainly from the inertia of the electric charges constituting the conduction and convection currents; and S is the vector determining the flow of electromagnetic energy outwards over the bounding surface.

In this equation it is generally assumed that our knowledge of the forms of F and W is precise and accurate, and that in fact in agreement with the results of paragraph 8.

$$W = \frac{1}{8\pi}\int E^2\,dv+\int dv\int^P (E'\,\delta P)$$

where P is the polarisation intensity of the dielectric media produced by the electromotive force

$$E' = E+\frac{1}{c}[V_m B]$$

whilst

$$F = \int (EC_1)\,dv$$

where C_1 represents the part of the total current depending on the motion of the electrons constituting the conduction current and the current due to the convection of electric charges, but excluding the part due to the convection of the polarised media.

It seems difficult to dispute the form of the expression for W, but careful consideration will also convince one that it is probably just as difficult to support it in the most general case, except it be by the results which are derived from it, which certainly seem to be in satisfactory agreement with our knowledge of these things. A similar reservation must be applied also to the expression for F, but there is here an additional point worth noticing. It is not often remarked that the form given tacitly involves an assumption which is derived as an independent result from discussions based on this special form. In fact it involves the definite assumption that no work is done by the magnetic forces during the motion of electric charges. Of course the usual expression for such force as proportional to the vector product of the velocity and magnetic force confirms this assumption, but the derivation of this expression by dynamical methods from results derived from the present discussion is by so much deprived of interest. In fact, if to the assumption that these forces do no work we add the further conditions that they are linear in the magnetic and velocity vectors, it would appear that their form is completely determined, at least to a constant factor, without further considerations either of a dynamical or any other nature.

The form for the expression T is not usually regarded as being sufficiently definite to be used in the present connexion, mainly because it is the more readily convertible into equally simple alternative forms. We have in our previous discussions made certain assumptions which have proved to be equivalent to taking

$$T = \frac{1}{4\pi}\int dv \int^{B} (H\, dB),$$

but this special form will subsequently be proved to be irrelevant to the discussion. It is usually regarded as being most advisable to consider the expression for T as bound up with that for S, the equation connecting the two being that derived from the energy principle by the insertion of the forms chosen for W and F, viz.,

$$\frac{dT}{dt} + \int_f S_n\, df = -\int (EC)\, dv - \int \frac{1}{c}\left([\nu_m B],\ \frac{dP}{dt} \right) dv + \frac{1}{c}\int (E,\ \text{Curl}\, [P\nu_m]\,)\, dv$$

C being now the total current of Maxwell's theory. This is all we can derive from the energy principle. The various possibilities open to us have been examined in detail before. We may take

$$\text{Curl}\, H = \frac{4\pi}{c}\, C$$

and then we get POYNTING's theory in which

$$T = \frac{1}{4\pi}\int dv \left[\int^{B} (H - \frac{4\pi}{c}[P\nu_m], dB) - \frac{1}{c}\int^{P} ([\nu_m B]\, dP) \right]$$

and

$$S = \frac{c}{4\pi}[E, H - 4\pi[P\nu_m]] = \frac{c}{4\pi}[E, H']$$

where

$$H' = H - 4\pi[P\nu_m]$$

is equivalent to the vector H′ introduced in paragraph 8.

This theory has the advantage, in addition to that already discussed at length, that it involves no further dynamical assumption other than those expressed in the special forms chosen for W and F. The Amperean equation used to effect the separation being more in the nature of a kinematical definition of the electric current or magnetic force than of a dynamical relation between the field vectors.

Another form can be obtained by using the equations

$$E = -\frac{1}{c}\frac{dA}{dt} - \text{grad}\,\phi$$

with

$$\text{div}\, C = 0$$

we then get

$$T = \frac{1}{c}\int dv \int^{A} (C\, dA) - \frac{1}{c}\int dv \int^{P} (\nu_m [B\, dP])$$
$$- c\int^{A} (\text{Curl}\,[P\nu_m], dA)$$

with

$$S = \phi\,(C - c\,\text{Curl}\,[P\nu_m]).$$

The special form of this result when the media are at rest has been shown* to be inconsistent with our usual conception of such things as radiation phenomena.

Yet another form of the theory can be obtained by taking

$$T = -\int dt \int \left[(EC + \frac{1}{c}\left([\nu_m B], \frac{dP}{dt}\right) - c\,(E\ \text{Curl}\,[P\nu_m]) \right] dv$$

and then

$$S = 0.$$

In such a theory there would be no such thing as radiation.

We can go on multiplying the different forms of this theory indefinitely and each form obtained would in itself be perfectly consistent with the Maxwellian electrodynamic theory. The expressions for S and T in them are of course dependent

* 'Phil. Mag.,' vol. 34 (1917), p. 385.

on each other, being connected by the equation given above, and there will be a relation between the forms for two different theories. In fact if S and T are the forms corresponding to any one mode of separation and if we write

$$S' = S + \frac{dU}{dt}$$

where U is any arbitrary vector function we shall have

$$\frac{dT}{dt} + \int S_n \, df = \frac{dT}{dt} + \int \left(S'_n - \frac{dU_n}{dt} \right) df$$
$$= \frac{dT'}{dt} + \int S'_n \, df$$

where

$$T' = T - \int \text{div } U dv,$$

and S′ and T′ are appropriate forms for a new mode of separation. In this way, by assigning convenient values for U, we might tentatively construct a number of interesting formulæ.

The last result also shows why it is that the particular form chosen for the kinetic energy is irrelevant to the general dynamical discussion of paragraph 7. In fact, if, instead of the form T used on that occasion, we had employed the general value derived above

$$T - \int \text{div } U \, dv$$

the part of the variation depending on this energy becomes the time integral of

$$\delta T - \int \text{div } \delta U \, dv,$$

and the latter integral reduces to a surface integral over the infinitely distant boundary and cannot therefore contribute anything in this general variational equation.

Of course, from another point of view, the various forms of the theory here under review, differ merely in assigning different distributions to the magnetic energy in the field, each of these distributions being ultimately consistent with the same proper total for this quantity; and the fact that they all lead to the same dynamical equations, merely verifies a well-known result of analytical dynamics that the particular form of expression for the energies of the system is immaterial to the ultimate dynamical equations for the field inside a continuous medium. Of course the solutions of boundary problems such as are, for example, involved in a specification of the energy flux, depend essentially on the particular form assumed for the energy distribution;

and it has been shown on a previous occasion* that the only form of specification of the energy distribution which is consistent with our usual ideas on these matters is that which makes the density of the magnetic energy equal to $B^2/8\pi$, and as the present discussion shows that our only hope of discrimination lies in that direction, we may assume that the evidence in favour of this special form is conclusive, at least for the present; it is besides the only form in which the most general case is completely representative of the distribution in any volume of the field without requiring the introduction of boundary terms involving surface distributions.

11. We next turn to a consideration of the expression for the forcive of electromagnetic origin acting on the polarised media in the field. We have seen that the mechanical forcive on the dielectrically polarised media is such that its x-component per unit volume at any place is of the form

$$(P\nabla)\,E_x+\frac{1}{c}\left([P\nu_m],\ \frac{\partial B}{\partial x}\right)+\frac{1}{c}\left[\frac{dP}{dt},\ B\right]_x$$

or as it first appears in the analysis

$$\left(P\frac{\partial E}{\partial x}\right)-[P\ \mathrm{Curl}\ E]_x+\frac{1}{c}\left[\frac{dP}{dt},\ B\right]+\frac{1}{c}\left([P\nu_m]\,\frac{\partial B}{\partial x}\right)$$
$$=P\frac{\partial E}{\partial x}+\frac{1}{c}\frac{d}{dt}[PB]+\frac{1}{c}\left([P\nu_m]\,\frac{\partial B}{\partial x}\right).$$

This result is in complete agreement with that derived by LARMOR in the electron theory,† but the present derivation indicates clearly the origin of the different terms in it. The expression

$$\left(P\frac{\partial E}{\partial x}\right)$$

is that corresponding to the expression derived in the statical theory from energy considerations and corresponds to MAXWELL'S magnetic expression; the second term, viz.,

$$-[P\ \mathrm{Curl}\ E]_x$$

is one of the terms arising as a result of the convection of the media, and this is the term which is effective in reducing the electric part of the forcive to the form

$$(P\nabla)\,E_x$$

which is the result derived in the elementary theory by regarding the forcive as the resultant of the forces on the elementary bi-poles.

* 'Phil. Mag.,' vol. 34 (1917), p. 385. *Cf.* also 'Phil. Mag.,' vol. 32 (1916), p. 162.

† 'Æther and Matter,' p. 104.

Similar results apply in the magnetic case. The general expression for the forcive on the magnetic media is, per unit volume, equal to

$$\left(\mathrm{I}\frac{\partial \mathrm{B}}{\partial x}\right)-[\mathrm{I}\ \mathrm{Curl}\ \mathrm{B}]_x-\frac{1}{c}\left([\mathrm{I}\nu_m],\ \frac{\partial(\mathrm{E}+\mathrm{E}_0)}{\partial x}\right)$$

$$-\frac{1}{c}\left[\frac{d\mathrm{I}}{dt},\ \mathrm{E}+\mathrm{E}_0\right]_x$$

$$=(\mathrm{I}\nabla)\,\mathrm{B}_x-\frac{1}{c}\left([\mathrm{I}\nu_m]\frac{\partial(\mathrm{E}+\mathrm{E}_0)}{\partial x}\right)-\frac{1}{c}\left[\frac{d\mathrm{I}}{dt},\ \mathrm{E}+\mathrm{E}_0\right]$$

$$=\left(\mathrm{I}\frac{\partial \mathrm{B}}{\partial x}\right)-\frac{1}{c}\frac{d}{dt}[\mathrm{I},\ \mathrm{E}+\mathrm{E}_0]-\frac{1}{c}\left([\mathrm{I}\nu_m],\ \frac{\partial(\mathrm{E}+\mathrm{E}_0)}{\partial x}\right).$$

In these expressions the parts depending on E_0 and I^2, representing as they do forces on the elements of the media determined solely by the conditions in those elements, would be neglected in a mechanical theory.* The expression for the effective forcive thus reduces to

$$\left(\mathrm{I}\frac{\partial \mathrm{H}}{\partial x}\right)-\frac{1}{c}\frac{d}{dt}[\mathrm{IE}]-\frac{1}{c}\left([\mathrm{I}\nu_m],\ \frac{\partial \mathrm{E}}{\partial x}\right)$$

$$=(\mathrm{I}\nabla)\,\mathrm{H}-\frac{1}{c}\left([\mathrm{I}\nu_m]\frac{\partial \mathrm{E}}{\partial x}\right)-\frac{1}{c}\left[\frac{d\mathrm{I}}{dt},\ \mathrm{E}\right].$$

This expression is only equivalent to MAXWELL's expression in the statical case he considers. It is, however, practically equivalent to that derived by counting the forces on the constituent poles, but even here the general result rather suggests a modified conception of the force on a magnetic pole, this force involving in the general case a term due to the electric force. The question of the existence of forces on a magnetic pole due to its motion in an electric field does not appear to have been investigated on an independent basis, although it is definitely contained in the relations of transformation involved in the theory of relativity, which require the form for this forcive

$$\mathrm{B}-\frac{1}{c}[\nu_m\mathrm{E}].$$

It will however be proved below in the next paragraph that such forces do probably exist and are in fact of precisely the correct type.

It may, of course, be objected that the last term in the equation

$$\frac{d\mathrm{B}}{dt}=\frac{d\mathrm{H}}{dt}+4\pi\frac{d\mathrm{I}}{dt}+4\pi\ \mathrm{Curl}\ [\mathrm{I}\nu_m],$$

which is the origin of the discrepancy obtained for the magnetic forcive, does not in reality exist, but yet the other results derived from this equation are almost certainly

* *Cf.* LARMOR, 'Æther and Matter,' p. 98.

indisputable, and it seems difficult to realise a state of affairs where the equation without the last term would be generally true.

The whole question of mechanical forces on polarised media is ultimately bound up with the question of the variation of the intrinsic energy of those media, and the expression

$$\left(\mathrm{I}\frac{\partial \mathrm{H}}{\partial x}\right),$$

for the x-component of the forcive per unit volume implies that the internal energy of those media change in a small displacement by

$$(\mathrm{H}\ \delta\mathrm{I})$$

per unit volume. But when the media are in motion the expression for the change $\delta\mathrm{I}$ used in this expression involves a part due to the convection of the polarisation which is more properly concerned with the mechanical forces than with the intrinsic elastic or motional ones, as it would exist if the internal constitution of the media was maintained rigidly constant. It is therefore suggested that the result derived above, that the expression for the rate of change of the intrinsic energy is practically

$$\left(\mathrm{H}-\frac{1}{c}[\nu_m\mathrm{E}],\ \delta\mathrm{I}\right)$$

per unit volume, is the more legitimate form of this expression, as allowance is made in it for the convection, and if this is granted, then the equivalent expression for the mechanical forcive, viz.,

$$(\mathrm{I}\nabla)\,\mathrm{H}_x-\frac{1}{c}\left([\mathrm{I}\nu_m]\frac{\partial \mathrm{E}}{\partial x}\right)-\frac{1}{c}\left[\frac{d\mathrm{I}}{dt},\ \mathrm{E}\right]$$

must be regarded as the only adequate form.

Moreover these two expressions essentially involve the particular form of equation adopted for defining $d\mathrm{H}/dt$, and are the only ones which are capable of fitting in with a general relativity theory.

The results here derived also emphasise the difficulties involved in treating the currents due to the convection of polarised media as effectively equivalent to a polarisation of the opposite kind. If, for example, we had treated the convection current

$$\mathrm{Curl}\ [\mathrm{P}\nu_m]$$

as equivalent to a distribution of magnetic polarity of intensity

$$[\mathrm{P}\nu_m]$$

at each place from the outset we should have been led to an entirely erroneous expression for the forcive on the polarised media, the reason being that the inclusion of this quantity with the magnetism hides its true character, particularly as regards its dependence on the velocity of the medium.

Nevertheless many of the relations of the theory will be considerably simplified if this procedure is adopted.

12. The question of forces on fictitious magnetic poles moving in an electric field is easily resolved by imparting to such poles a substantiality sufficient to allow us to talk of forces on them, and then applying any of the general methods used in this theory. The Lagrangian function of the system is still of the form

$$\mathrm{L}+\frac{1}{8\pi}\int(\mathrm{B}^2-\mathrm{E}^2)\,dv,$$

L being that part of it which is not directly determined by the conditions in the field and which will as usual be assumed to be a function of the positions and velocities of the electric and magnetic elements only. The sequence of changes is then best described by the fact that the action integral

$$\int_{t_1}^{t_2} dt\left[\mathrm{L}+\frac{1}{8\pi}\int(\mathrm{B}^2-\mathrm{E}^2)\,dv\right]$$

taken between fixed time limits is stationary subject to the implied conditions of the field. If we assume generally that there are a number of discrete electric particles as well in the field, these conditions may be written in the form

$$\operatorname{div}\mathrm{E}-4\pi\Sigma e = 0,$$

$$\operatorname{div}(\mathrm{B}-\mathrm{H})+4\pi\Sigma m = 0,$$

$$\operatorname{Curl}\mathrm{H}-\frac{1}{c}\frac{d\mathrm{E}}{dt}-\frac{4\pi}{c}\Sigma e\nu_e = 0,$$

$$\frac{d\mathrm{B}}{dt}-\frac{d\mathrm{H}}{dt}-4\pi\Sigma m\nu_m = 0,$$

wherein e is the charge of the typical electron and ν_e its velocity, m is the strength of the typical magnetic particle whose velocity is ν_m and the sum Σ in each equation is taken per unit volume at each place over the respective elements indicated in it.

We now introduce four undetermined Lagrangian multiplying functions, two scalar quantities ϕ_1 and ϕ_2, and two vector quantities A_1 and A_2, it is thus the variation of

$$\int dt\left[\mathrm{L}+\frac{1}{8\pi}\int dv\left[\mathrm{B}^2-\mathrm{E}^2+2\phi_1\operatorname{div}\mathrm{E}+2\phi_2\operatorname{div}(\mathrm{B}-\mathrm{H})\right.\right.$$
$$\left.-2\left(\mathrm{A}_1,\operatorname{Curl}\mathrm{H}-\frac{1}{c}\frac{d\mathrm{E}}{dt}\right)+\frac{2}{c}\left(\mathrm{A}_2\frac{d\mathrm{B}}{dt}-\frac{d\mathrm{H}}{dt}\right)\right]$$
$$\left.-8\pi\Sigma e\phi_1+8\pi\Sigma m\phi_2+\frac{8\pi}{c}\Sigma e(\mathrm{A}_1\nu_e)-\Sigma m(\mathrm{A}_2\nu_m)\right],$$

that is to be made null, afterwards determining the functions ϕ_1, ϕ_2, A_1, A_2 to satisfy the restrictions which necessitated their introduction. The variation can now be affected in the usual way, and the condition that it vanishes leads to the following equations

$$E+\text{grad}\ \phi_1+\frac{1}{c}\frac{dA_1}{dt}=0,$$

$$B-\text{grad}\ \phi_2-\frac{1}{c}\frac{dA_2}{dt}=0,$$

with

$$\text{Curl}\ A_1-\text{grad}\ \phi_2-\frac{1}{c}\frac{dA_2}{dt}=0,$$

with three equations of each of the types

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_e}\right)-\frac{\partial L}{\partial x_e}=-e\left(\frac{\partial \phi_1}{\partial x_e}+\frac{1}{c}\frac{dA_{1x}}{dt}\right)+\frac{e}{c}[\nu_e\ \text{Curl}\ A_1],$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_m}\right)-\frac{\partial L}{\partial x_m}=m\left(\frac{\partial \phi_2}{\partial x_m}+\frac{1}{c}\frac{dA_{2x}}{dt}\right)-\frac{m}{c}[\nu_m\ \text{Curl}\ A_2].$$

The first and third of these equations show that ϕ_1 and A_1 are the usual scalar and vector potentials; in fact from the third we have

$$\text{Curl}\ A_1=\text{grad}\ \phi_2+\frac{1}{c}\frac{dA_2}{dt}$$
$$=B,$$

so that A_1 is the vector potential and then ϕ_1 is the scalar potential.

The fourth equation thus determines the usual expression for the reaction forces on the moving electron; the fifth equation determines similarly the force on the moving magnetic pole in the form

$$m\left(\text{grad}\ \phi_2+\frac{1}{c}\frac{dA_2}{dt}\right)-\frac{m}{c}[\nu_m\ \text{Curl}\ A_2]=m\left(B-\frac{1}{c}[\nu_m,\ \text{Curl}\ A_2]\right).$$

We have yet to determine Curl A_2: we have

$$B=\text{grad}\ \phi_2+\frac{1}{c}\frac{dA_2}{dt},$$

so that

$$\text{Curl}\ B=\frac{1}{c}\frac{d}{dt}\text{Curl}\ A_2.$$

Now

$$\text{Curl}\ B=\frac{4\pi}{c}C_1+4\pi\ \text{Curl}\ I+\frac{1}{c}\frac{dE}{dt}=\frac{4\pi}{c}C_0=\frac{1}{c}\frac{dE}{dt},$$

wherein C_1 is the true current density of electric flux;

$$I = \frac{B-H}{4\pi},$$

is the intensity of magnetic polarity, and

$$C_0 \equiv C_1 + c \text{ Curl } I.$$

It follows that

$$\text{Curl } A_2 = E + 4\pi \int^t C_0 \, dt = E + E_0,$$

say. The vector A_2 is a slightly more general form of the second vector potential introduced in our previous dynamical discussion and its *curl* is identical with the *curl* of that vector.

The main part of Curl A_2 is thus determined by the electric force in the field, and its mechanically effective part is completely represented by this vector; the forcive on the moving magnetic pole is thus to all intents and purposes equal to

$$m\left(B - \frac{1}{c}[\nu_m E]\right)$$

an expression which agrees with that suggested by the relativity transformation.

It must, however, be noted that the local term is necessary in the complete relation defining the vector A_2 for the simpler relation

$$\text{Curl } A_2 = E,$$

carries with it the consequence that

$$\text{div } E = 0$$

at all points of the field, and this is true only of those points where there is no electricity.

The expression for the forcive on the magnetic media is now attainable by regarding it as the resultant of the forces on its contained poles; for the volume v bounded by the closed surface f, it is in fact

$$-\int (\text{div } I) \, B \, dv + \int I_m B \, df$$

$$+ \frac{1}{c}\int \left[\frac{dI}{dt} + c \text{ Curl } [I\nu_m], \; E + E_0\right] dv$$

$$+ \int [[[I\nu_m] \, n_1], \; E + E_0] \, df.$$

The second and fourth integrals transform by GREEN's lemma to the volume integrals

$$\int (B(\nabla I) + (I\nabla) B) \, dv - \int [[\text{Curl } [I\nu_m] \, E + E_0] + \text{grad } ([I\nu_m] \, E + E_0)$$

$$+ \qquad -([I\nu_m] \text{ div } (E + E_0))] \, dv,$$

where in the last term but one the gradient operation only affects the E functions. Now

$$\operatorname{div}(\mathrm{E}+\mathrm{E}_0) = 0,$$

and thus the resultant force may be taken as distributed throughout the volume with intensity at each point equal to

$$(\mathrm{I}\nabla)\,\mathrm{B}+\frac{1}{c}\left[\frac{d\mathrm{I}}{dt},\ \mathrm{E}+\mathrm{E}_0\right]-\operatorname{grad}\left(\,[\mathrm{I}\nu_m],\ \mathrm{E}+\mathrm{E}_0\right)$$

in agreement with the general result derived above. The local terms in I^2 and E_0 may again be presumed to balance out with other forces of a type not at present under review.

13. The two new potentials A_2 and ϕ_2, introduced in the analysis of the last paragraph, are the general forms of the potentials analogous to the ordinary scalar and vector potentials of this theory, and they satisfy similar equations. We have, in fact,

$$\operatorname{Curl}\mathrm{A}_2 = \mathrm{E}+4\pi\int^t \mathrm{C}_0\,dt$$

where C_0 is the total current density of electric flux including the effective representation of the magnetism. Thus

$$\operatorname{Curl}\operatorname{Curl}\mathrm{A}_2 = \operatorname{Curl}\mathrm{E}+4\pi\int^t \operatorname{Curl}\mathrm{C}_0\,dt$$

$$= -\frac{1}{c}\frac{d\mathrm{B}}{dt}+4\pi\int^t \operatorname{Curl}\mathrm{C}_0\,dt.$$

Thus

$$\operatorname{grad}\operatorname{div}\mathrm{A}_2-\nabla^2\mathrm{A}_2 = -\frac{1}{c^2}\frac{d^2\mathrm{A}_2}{dt^2}-\frac{1}{c}\operatorname{grad}\frac{d\phi_2}{dt}+4\pi\int^t \operatorname{Curl}\mathrm{C}_0\,dt$$

whilst since $\operatorname{div}\mathrm{B} = 0$, we have also

$$\nabla^2\phi_2+\frac{1}{c}\frac{d}{dt}\operatorname{div}\mathrm{A}_2 = 0.$$

We may now adopt one of a number of alternatives. The simplest one is got by taking $\phi_2 = 0$, when we also have

$$\operatorname{div}\mathrm{A}_2 = 0$$

with, therefore,

$$\nabla^2\mathrm{A}_2 = \frac{1}{c^2}\frac{d^2\mathrm{A}_2}{dt^2}+4\pi\int^t \operatorname{Curl}\mathrm{C}_0\,dt.$$

The last equation really involves the first, for

$$\nabla^2(\operatorname{div}\mathrm{A}_2) = \frac{1}{c^2}\frac{d^2}{dt^2}(\operatorname{div}\mathrm{A}_2),$$

so that $\operatorname{div}\mathrm{A}_2$ must be zero as it has no singularities.

The vector A_2 chosen in this way is, practically speaking, the æthereal displacement vector employed by LARMOR in his mechanical model of the electric and luminiferous medium. The *curl* of this vector is the electric force, or at least as regards its rate of change, whilst the magnetic induction B, which is proportional to the time rate of change of A_2, appears as the velocity.

We need not, however, take the quantities in this way. We might take

$$\text{div } A_2 = \frac{1}{c}\frac{d\phi_2}{dt} - 4\pi \text{ div} \int^t I\, dt,$$

and then we should have

$$\nabla^2\phi_2 = \frac{1}{c^2}\frac{d^2\phi_2}{dt} - \text{div } I$$

with

$$\nabla^2 A'_2 = \frac{1}{c^2}\frac{d^2 A'_2}{dt^2} + 4\pi\frac{dI}{dt} + 4\pi\int^t \text{Curl } C_0\, dt$$

where we have used

$$A'_2 = A_2 - 4\pi\int^t I\, dt.$$

In this case ϕ_2 is the scalar potential of the magnetic distribution, whilst A'_2 belongs to the current distribution. With these differential equations the general values of ϕ_2 and A_2 in regular fields are such that

$$\phi_2 = \frac{4\pi}{c}\int\frac{[\text{div } I]}{r}\, dv$$

whilst

$$\frac{1}{c}\frac{dA_2}{dt} - \frac{4\pi I}{c} = \frac{4\pi}{c^2}\int r^{-1}\left[\text{Curl } C_0 + \frac{d^2 I}{dt^2}\right] dv$$

the square brackets in the integrands denoting that their values are taken at each point for the time $\left(t - \frac{r}{c}\right)$.

These are the most interesting cases of the solutions for ϕ_2 and A_2, but we may construct any number of others. It must be noticed, however, that the equation

$$B = \frac{1}{c}\frac{dA_2}{dt} + \text{grad } \phi_2,$$

does not imply that the vector B is derived from a potential in steady fields, for it is impossible to satisfy the equations with A_2 independent of the time; we may have dA_2/dt constant in time but not A_2. This is the origin of the difficulty in LARMOR's mechanical model which seems to necessitate the piling up of æthereal displacement in a steady magnetic field.

14. We have determined the complete expression for the forcive per unit volume on the media occupying the electromagnetic field. The next step in the general

theory is to reduce these forces to a representation by means of an applied stress system of ordinary character. This discussion leads in the usual way to the introduction of the concept of electromagnetic momentum.

The actual calculations for the present form of the results are not materially different from those given *in extenso* elsewhere, so that it will again be sufficient to outline the principal stages in the discussion. The method employed is to attempt to express, say, the x component of the force per unit volume in the form

$$\frac{\partial T_{xx}}{\partial x}+\frac{\partial T_{xy}}{\partial y}+\frac{\partial T_{xz}}{\partial z}.$$

Now the total forcive of electrodynamic origin acting on the medium of the field at any place is such that its x component per unit volume is

$$\left(P-\frac{1}{c}[I\dot{r}_m],\ \frac{\partial E}{\partial x}\right)+\left(I+\frac{1}{c}[P\dot{r}_m],\ \frac{\partial B}{\partial x}\right)$$
$$-[P,\ \text{Curl E}]_x-[I,\ \text{Curl B}]_x-\frac{1}{c}\left[B,\frac{dP}{dt}\right]+\frac{1}{c}\left[E,\frac{dI}{dt}\right]$$
$$+\rho\left(E_x+\frac{1}{c}[\nu_m B]\right)+\frac{1}{c}[C_1 B]_x.$$

Again writing

$$H' = H-\frac{4\pi}{c}[P\nu_m],$$

it is proved just as in the usual form of the theory that the forcive of which this is the representative component is represented in the main by a stress system in which

$$T_{xx} = E_x D_x-\frac{1}{8\pi}E^2+\frac{1}{4\pi}B_x H'_x-\frac{1}{8\pi}H'^2$$

and

$$T_{xy} E_x D_y+\frac{1}{4\pi}B_{xy}\ H'_x$$

with symmetrical expressions for the other constituents; but with this representation there is an outstanding part of the complete forcive, viz.,

$$-\frac{1}{4c}\frac{d}{dt}[EB]+\frac{1}{c}\frac{d}{dt}[EI]-\frac{1}{c}\left([I\nu_m],\frac{\partial E}{\partial x}\right) = -\frac{1}{4\pi c}\frac{d}{dt}[EH]-\frac{1}{c}\left([I\dot{\nu}_m],\frac{\partial E}{\partial x}\right)$$

which cannot be so reduced. The first term in this outstanding part, being a complete differential with respect to the time, is usually taken to represent a part of the complete forcive arising as the kinetic reaction to a rate of change of momentum, and this is the origin of the concept of electromagnetic momentum. This idea is however partly destroyed by the remaining term in the above expression which cannot be developed either as a forcive of ordinary type or as a kinetic reaction to a rate of change of momentum, so that we are rather forced to regard these outstanding terms as pointing to the failure of the ideas from which we set out. This conclusion does not, of course, invalidate the results derived in the simpler electron theory, as the concept of momentum will remain under the simplest conditions as a convenient mathematical expression for the actual result, whatever be its ultimate physical basis.

The present formulation possesses another disadvantage which is apparently not inherent in the simplest presentations of the momentum idea. In the electron theory, as usually developed, the momentum remains as a fundamental quantity and is distributed over the field with the density

$$\frac{1}{4\pi c}[\mathrm{EB}]$$

at each point; this gives it a purely æthereal constitution as the vectors E and B are those which define the conditions in the æthereal field. In the present formulation the vector B is replaced by the vector H which is essentially an auxiliary mechanical vector in the theory; the fundamental nature of the momentum vector is therefore entirely lost. We can, of course, assume that some of the momentum is in reality attached to the matter, and such an assumption has certain points in its favour. The force of electromagnetic origin on the dielectric media for example has an x component which per unit volume is

$$\left(\mathrm{P}\frac{\partial \mathrm{E}}{\partial x}\right)+\frac{1}{c}\left([\mathrm{P}\nu_m],\frac{\partial \mathrm{B}}{\partial x}\right)-[\mathrm{P}\ \mathrm{Curl}\ \mathrm{E}]_x+\frac{1}{c}\left[\frac{d\mathrm{P}}{dt},\mathrm{B}\right],$$

and this may be written in the form

$$\left(\mathrm{P}\frac{\partial \mathrm{E}}{\partial x}\right)+\frac{1}{c}\left([\mathrm{P}\nu_m],\frac{\partial \mathrm{B}}{\partial x}\right)+\frac{1}{c}\frac{d}{dt}[\mathrm{PB}].$$

The first two terms appear as those appropriate to the energy function in the statical theory which would be

$$-(\mathrm{PE})-\frac{1}{c}([\mathrm{P}\nu_m],\mathrm{B}),$$

so that the third might be regarded as a kinetic reaction to a rate of change of momentum, which would be distributed throughout the medium with a density

$$\frac{1}{c}[\mathrm{PB}]$$

at each point.

A similar analysis and analogous results hold for the magnetic media.

There is, too, a relation satisfied by the momentum vector which appears in the simplest form of the theory and to which a fundamental significance is attached by some authors, but which is not satisfied by the results of our present discussion. The vector determining the flux of electromagnetic energy has been seen to be

$$S = \frac{c}{4\pi}[E, H']$$

and that determining the momentum is

$$M = \frac{1}{4\pi c}[E, H].$$

In the absence of magnetic media and convective dielectric polarisations these two expressions satisfy the equation

$$M = \frac{1}{c^2}S,$$

but under the most general circumstances this relation is not satisfied.

We have so far conducted the discussions as though the quantity derived as a momentum is unique and definite, whereas, as a matter of fact, this is far from being the case. We saw that the idea of the momentum arose from certain outstanding terms which remained when attempting to reduce the electromotive forces to a representation by a stress system. Now we can give a number of different forms to this reduction and each one carries with it a different expression for the electromagnetic momentum. We can, for instance, write

$$\frac{1}{4\pi c}[EH] = -\frac{1}{4\pi c^2}\left[\frac{dA}{dt}, H\right] - \frac{1}{4\pi c}[\nabla\phi, H]$$
$$= -\frac{1}{4\pi c^2}\left[\frac{dA}{dt}, H\right] - \frac{1}{4\pi c}[\nabla\phi, H] + \frac{1}{c_2}\phi C$$

and the second term in this expression when differentiated with respect to the time might be included in the stress specification. This would leave a new expression for the electromagnetic momentum which is

$$\frac{1}{c_2}\left(\phi C - \frac{1}{4\pi}\left[\frac{dA}{dt}, H\right]\right)$$

a form which would probably be suitable for use in connexion with a theory in which the radiation phenomena are represented by MACDONALD'S form of the theory.

This is not the only alternative to the usual theory for we can construct similarly any number of others. It appears, however, that the usual presentation is probably the simplest possible one, and this is a great advantage in its favour; but subsequent developments of the theory may require a modification, and then it is as well to remember that there are other forms of the theory perfectly consistent with the general relations of the electromagnetic field, both as regards its general and dynamical aspects.

It is hoped in a future communication to examine in detail some of these alternative expressions for the momentum, but so far the results obtained are not of sufficient interest to warrant their discussion at the present stage.

15. It may now be convenient to summarise the conclusions and results of our discussion. The differential theory of MAXWELL as expressed in the usual way by the equations

$$\text{Curl H} = \frac{4\pi}{c}\left(\frac{d\text{D}}{dt} + \text{C}_1 - \rho\nu\right) \qquad \text{Curl E} = -\frac{1}{c}\frac{d\text{B}}{dt} \qquad \text{div D} = \rho$$

is supplemented by the addition of an equation expressing the rate of change of the magnetic force

$$\frac{d\text{H}}{dt} = \frac{d\text{B}}{dt} - 4\pi\frac{d\text{I}}{dt} - 4\pi\,\text{Curl}\,[\text{I}\nu_m]$$

this equation being analogous to that expressing the rate of change of electric displacement

$$\frac{d\text{D}}{dt} = \frac{1}{4\pi}\frac{d\text{E}}{dt} + \frac{d\text{P}}{dt} + \text{Curl}\,[\text{P}\nu_m].$$

The fundamental dynamical equations are then derived by a variational principle equivalent to the principle of Least Action in dynamics; in this discussion the assumption of a definite electronic constitution for the dielectric and magnetic polarisations is specially avoided in order to bring out certain points of the theory which have not previously received adequate treatment. In this way, in addition to deriving the equation

$$\text{Curl E} = -\frac{1}{c}\frac{d\text{B}}{dt},$$

it can be proved that the forces on the media occupying the field consist of several distinct parts. There is firstly a part

$$\rho\left(\text{F} + \frac{1}{c}[\nu_m\text{B}]\right)$$

due to the free charges associated with the typical point of the matter and a part

$$\frac{1}{c}[\text{C, B}]$$

due to the true conduction current. Due to the dielectric polarisations there is a part whose x component is

$$(\text{P}\nabla)\,\text{E}_x + \frac{1}{c}\left[\frac{d\text{P}}{dt},\ \text{B}\right]_x - \frac{1}{c}\left([\text{P}\nu_m]\,\frac{\partial\text{B}}{\partial x}\right),$$

and the magnetic polarisations give rise to the analogous terms

$$(\mathrm{IV})\ \mathrm{B}_x - \frac{1}{c}\left[\frac{d\mathrm{I}}{dt}, \mathrm{E}\right] - \frac{1}{c}\left([\mathrm{I}\nu_m], \frac{\partial \mathrm{E}}{\partial x}\right).$$

With the exception of the magnetic terms these results are in general agreement with those usually derived on the basis of the electronic theory, and the discrepancy in the magnetic terms is proved to arise from the inadequacy of the treatment of the magnetic relations in that theory, no allowance being made in it for the convection of the magnetic polarisations.

The results derived from the dynamical theory are then examined in connexion with the usual developments of the theory in regard to radiation phenomena, to the energetic relations of the magnetic media and, finally, to the fundamental problem of the representation of the forces in the field, as an applied stress system and the subsidiary question of electromagnetic momentum. In regard to each of these results are derived which do not differ materially from those usually given, but the slight discrepancies in each case, although probably of little or no practical significance, prove ultimately to be of theoretical importance as helping to justify the fundamental equations on which they are based. The auxiliary conception of electromagnetic momentum is not however completely attained under the most general conditions, although it will still remain to enable us to obtain an effective mode of expressing certain results of the simpler theory; it is probably present in no other capacity in former interpretations of the theory so that this is hardly a disadvantage of the present formulation.

The present theoretical relations require, of course, to be supplemented by the usual empirical laws for the induction of the two polarisations and the conduction current. We have however specially refrained from introducing these relations as it was desired to emphasise the fact that the theory in its complete form is entirely independent of these laws, so that for example it necessarily covers the most complex fields, involving ferromagnetic inductions and polarisations. If we interpret the theory as determining the electrodynamic changes in the system during its transition from one configuration to another even the presence of hysteritic qualities in the inductions will not vitiate its validity. This is, of course, no special advantage attaching to the present form of the theory as it is in reality fundamentally inherent in every interpretation, although it may be hidden by the particular form of expression adopted.